SOLUTIONS OF AN ELLIPTIC SYSTEM WITH A NEARLY CRITICAL EXPONENT

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ABSTRACT. Consider the problem

$$\begin{aligned} -\Delta u_\epsilon &= v_\epsilon^p \quad v_\epsilon > 0 \quad \text{in} \quad \Omega, \\ -\Delta v_\epsilon &= u_\epsilon^{q_\epsilon} \quad u_\epsilon > 0 \quad \text{in} \quad \Omega, \\ u_\epsilon &= v_\epsilon = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N > 2$, with smooth boundary $\partial\Omega$. Here $p, q_\epsilon > 0$, and

$$\epsilon := \frac{N}{p+1} + \frac{N}{q_\epsilon + 1} - (N-2).$$

This problem has positive solutions for $\epsilon > 0$ (with $pq_\epsilon > 1$) and no non-trivial solution for $\epsilon \leq 0$. We study the asymptotic behaviour of *least energy* solutions as $\epsilon \to 0^+$. These solutions are shown to blow-up at exactly one point, and the location of this point is characterized. In addition, the shape and exact rates for blowing up are given.

RÉSUMÉ. Considéré le problème

$$\begin{aligned} -\Delta u_\epsilon &= v_\epsilon^p \quad v_\epsilon > 0 \quad \text{en} \quad \Omega, \\ -\Delta v_\epsilon &= u_\epsilon^{q_\epsilon} \quad u_\epsilon > 0 \quad \text{en} \quad \Omega, \\ u_\epsilon &= v_\epsilon = 0 \quad \text{sur} \quad \partial\Omega, \end{aligned}$$

où Ω est un domaine convexe et borné de $\mathbb{R}^N$, $N > 2$, avec la frontière régulière $\partial\Omega$. Ici $p, q_\epsilon > 0$, et

$$\epsilon := \frac{N}{p+1} + \frac{N}{q_\epsilon + 1} - (N-2).$$

Ce problème a les solutions positives pour $\epsilon > 0$ (avec $pq_\epsilon > 1$) et non pas de solution non-trivial pour $\epsilon \leq 0$. Nous étudions le comportement asymptotique de solutions d'*énergie minimale* quand $\epsilon \to 0^+$. Ces solutions explosent en un seul point, et la localisation de ce point est characterisé. De plus, la forme et le rythme d'explosion sont donnés.

1. INTRODUCTION

We consider the elliptic system

$$-\Delta u_\epsilon = v_\epsilon^p \quad v_\epsilon > 0 \quad \text{in} \quad \Omega, \tag{1.1}$$
$$-\Delta v_\epsilon = u_\epsilon^{q_\epsilon} \quad u_\epsilon > 0 \quad \text{in} \quad \Omega, \tag{1.2}$$
$$u_\epsilon = v_\epsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary,—–, Secondary,—–.
Key words and phrases. Semilinear elliptic system, asymptotic behaviour, critical exponent.
This research was supported by FONDECYT 3040059.

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N > 2$, with smooth boundary $\partial\Omega$. Here $p, q_\epsilon > 0$, and

$$\epsilon := \frac{N}{p+1} + \frac{N}{q_\epsilon + 1} - (N-2). \tag{1.4}$$

When $\epsilon \leq 0$, there is no solution for (1.1)-(1.3), see [18] and [22]. On the other hand when $\epsilon > 0$, we can prove existence of solutions obtained by the variational method. In fact, for $\epsilon > 0$, the embedding $W^{2,\frac{p+1}{p}}(\Omega) \hookrightarrow L^{q_\epsilon+1}(\Omega)$ is compact for any $q_\epsilon + 1 > (p+1)/p$, that is $pq_\epsilon > 1$. Using this, it is not difficult to show that there exists a function $\bar{u}_\epsilon$ positive solution of the variational problem

$$S_\epsilon(\Omega) = \inf\left\{ \|\Delta u\|_{L^{\frac{p+1}{p}}(\Omega)} \;\mid\; u \in W^{2,\frac{p+1}{p}}(\Omega), \quad \|u\|_{L^{q_\epsilon+1}(\Omega)} = 1 \right\}, \tag{1.5}$$

see for example [23]. This solution satisfies $-\Delta\bar{u}_\epsilon = \bar{v}_\epsilon^p$, $-\Delta\bar{v}_\epsilon = S_\epsilon(\Omega)\bar{u}_\epsilon^{q_\epsilon}$, in Ω and $\bar{u}_\epsilon = \bar{v}_\epsilon = 0$ on $\partial\Omega$. After a suitable multiples of $\bar{u}_\epsilon$ and $\bar{v}_\epsilon$, we obtain u_ϵ and v_ϵ solving (1.1)-(1.3). We call (u_ϵ, v_ϵ) the *least energy solution* to (1.1)-(1.3). For others existence results, we refer to [4], [7], [9], [15], and [19].

Note that by setting $v_\epsilon = (-\Delta u_\epsilon)^{1/p}$, we can write the system (1.1)-(1.3) only in terms of u_ϵ, that is

$$-\Delta(-\Delta u_\epsilon)^{1/p} = u_\epsilon^{q_\epsilon} \quad u_\epsilon > 0 \quad \text{in} \quad \Omega \tag{1.6}$$

$$u_\epsilon = \Delta u_\epsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{1.7}$$

Concerning the least energy solutions, in [23] it was proved that $S_\epsilon(\Omega) \to S$ as $\epsilon \downarrow 0$, where S is independent of Ω and moreover is the best Sobolev constant for the inequality

$$\|u\|_{L^{q+1}(\mathbb{R}^N)} \leq S^{-\frac{p}{p+1}} \|\Delta u\|_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} \tag{1.8}$$

with p, q, N satisfying

$$\frac{N}{p+1} + \frac{N}{q+1} - (N-2) = 0. \tag{1.9}$$

This shows that the sequence $\{u_\epsilon\}_{\epsilon>0}$ of least energy solutions of (1.6)-(1.7) satisfy

$$S_\epsilon(\Omega) = \frac{\int_\Omega |\Delta u_\epsilon|^{\frac{p+1}{p}}\, dx}{\|u_\epsilon\|_{L^{q_\epsilon+1}(\Omega)}^{\frac{p+1}{p}}} = S + o(1) \quad \text{as} \quad \epsilon \to 0. \tag{1.10}$$

Relation (1.9) defines a curve in $\mathbb{R}_+^2$, for the variables p and q. This curve is the so-called *Sobolev Critical Hyperbola.* By symmetry, we assume without restriction that

$$2/(N-2) < p \leq p^* := (N+2)/(N-2). \tag{1.11}$$

For each fixed value of p, the strict inequality gives a lower bound for the dimension, i.e. $N > \max\{2, 2(p+1)/p\}$.

In this article, we shall study in detail the asymptotic behaviour of the variational solution u_ϵ, of (1.6)–(1.7) as $\epsilon \downarrow 0$, that is, as q_ϵ approaches from <u>below</u> to q given by the Sobolev Critical Hyperbola (1.9).

The asymptotic behaviour of the equation (1.6)-(1.7) as $\epsilon \downarrow 0$ has already been studied for $p = p^*$ and $p = 1$. Next we recall some of these results to introduce ours.

The case $p = p^*$ is equivalent to consider the single equation

$$-\Delta u_\epsilon = u_\epsilon^{p^*-\epsilon} \quad \text{in} \quad \Omega, \qquad \text{and} \quad u_\epsilon = 0 \quad \text{on} \quad \partial\Omega.$$

This problem was studied in [1, 10, 13, 20]. There, exact rates of blow-up were given and the location of blow-up points were characterized. One key ingredient was the Pohozaev identity and the observation that the solution u_ϵ, scaled in the form $\|u_\epsilon\|^{-1}_{L^\infty(\Omega)} u_\epsilon$ converges to U solution of

$$\begin{aligned} -\Delta U &= U^{p^*}, \quad U(y) > 0 \quad \text{for} \quad y \in \mathbb{R}^N && (1.12)\\ U(0) &= 1, \quad U \to 0, \quad \text{as} \quad |y| \to \infty, && (1.13)\end{aligned}$$

which is unique, explicit, and radially symmetric. For the location of blow-up and the shape of the solution away of the singularity, it was proved that a scaled u_ϵ, given by $\|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon$, converges to the Green's function G, solution of $-\Delta G(x, \cdot) = \delta_x$ in Ω, $G(x, \cdot) = 0$ on $\partial\Omega$. The location of blowing-up points are the critical points of $\phi(x) := g(x, x)$ (in fact their minima, see [10]), where $g(x, y)$ is the regular part of $G(x, y)$, i.e

$$g(x, y) = G(x, y) - \frac{1}{(N-2)\sigma_N |x-y|^{N-2}}.$$

In [6], a similar result was proven in the case $p = 1$, $(N > 4)$, where the problem is reduced to study (1.12)–(1.13) with the operator Δ^2 instead of $-\Delta$. Both cases give the blow-up rate

$$\epsilon \|u_\epsilon\|^2_{L^\infty(\Omega)} \to C \quad \text{as} \quad \epsilon \to 0.$$

for some explicit $C := C(p, N, \Omega) > 0$. We can ask ourselves if this behaviour is universal. We will see later that this is only a coincide.

Mimicking the above argument, we will study the asymptotic behaviour of the solution u_ϵ of (1.6)–(1.7) as $\epsilon \downarrow 0$. We shall show that $\|u_\epsilon\|^{-1}_{L^\infty(\Omega)} u_\epsilon$ converges, as $\epsilon \downarrow 0$, to the solution U of the problem

$$\begin{aligned} -\Delta U &= V^p, \quad V(y) > 0 \quad \text{for} \quad y \in \mathbb{R}^N && (1.14)\\ -\Delta V &= U^q, \quad U(y) > 0 \quad \text{for} \quad y \in \mathbb{R}^N && (1.15)\\ U(0) &= 1, \quad U \to 0, \quad V \to 0 \quad \text{as} \quad |y| \to \infty. && (1.16)\end{aligned}$$

In [5], it was proved that U and V are radially symmetric, if $p \geq 1$ and $U \in L^{q+1}(\mathbb{R}^N)$ and $V \in L^{p+1}(\mathbb{R}^N)$. This is the case when considering least energy solutions, see details in section 2. Thus $U(r) := U(y)$ and $V(r) := V(y)$ with $r = |y|$, moreover U and V are unique, and decreasing in r, see [16, 23]. There exist no explicit form of (U, V) for all $p \geq 1$, however to carry out the analysis it is sufficient to know the

asymptotic behaviour of (U,V) as $r\to\infty$, which was studied in [16]. They found

$$\lim_{r\to\infty} r^{N-2}V(r)=a \quad \text{and} \quad \begin{cases} \lim\limits_{r\to\infty} r^{N-2}U(r)=b & \text{if} \quad p>\frac{N}{N-2} \\ \lim\limits_{r\to\infty} \frac{r^{N-2}}{\log r}U(r)=b & \text{if} \quad p=\frac{N}{N-2} \\ \lim\limits_{r\to\infty} r^{p(N-2)-2}U(r)=b & \text{if} \quad \frac{2}{N-2}<p<\frac{N}{N-2}. \end{cases} \tag{1.17}$$

The aim of this paper is to show the following results.

Theorem 1.1. *Let u_ϵ be a least energy solution of (1.6)–(1.7) and $p\geq 1$. Then*
a) there exists $x_0\in\Omega$ such that, after passing to a subsequence, we have

$$\text{i)} \quad u_\epsilon\to 0\in C^1(\Omega\setminus\{x_0\}), \qquad \text{ii)} \quad v_\epsilon=|\Delta u_\epsilon|^{\frac{1}{p}}\to 0\in C^1(\Omega\setminus\{x_0\})$$

as $\epsilon\to 0$ and

$$\text{iii)} \quad |\Delta u_\epsilon|^{\frac{p+1}{p}}\to \|V\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}\delta_{x_0} \quad \textit{as} \quad \epsilon\to 0$$

in the sense of distributions.
b) x_0 is a critical point of

$$\begin{aligned} \phi(x) &:= g(x,x) \quad \textit{if} \quad p\in[N/(N-2),(N+2)/(N-2)) \quad \textit{and} &(1.18)\\ \tilde{\phi}(x) &:= \tilde{g}(x,x) \quad \textit{if} \quad p\in(2/(N-2),N/(N-2)) &(1.19)\end{aligned}$$

for $x\in\Omega$. The function $\tilde{g}(x,y)$ is defined for $p\in(2/(N-2),N/(N-2))$ by

$$\tilde{g}(x,y)=\tilde{G}(x,y)-\frac{1}{(p(N-2)-2)(N-p(N-2))(N-2)^p\sigma_N^p|x-y|^{p(N-2)-2}}$$

where $-\Delta\tilde{G}(x,\cdot)=G^p(x,\cdot)$ in Ω, $\tilde{G}(x,\cdot)=0$ on $\partial\Omega$.

We observe that regularity of $\tilde{\phi}$ is needed to compute its critical points in b). We show next that $\tilde{\phi}$ is regular. By definition of $\tilde{G}$, we have

$$\lim_{y\to x}|x-y|^{(p-1)(N-2)}\Delta\tilde{g}(x,y)=-\frac{pg(x,x)}{((N-2)\sigma_N)^{p-1}} \tag{1.20}$$

for $x\in\Omega$. Thus $-\Delta\tilde{g}(x,\cdot)\in L^q(\Omega)$ for any $q\in(N/2,N/(p(N-2)-N+2))$. This implies, by regularity, that $\tilde{g}(x,\cdot)\in L^\infty(\Omega)$ and therefore $\tilde{\phi}(x)=\tilde{g}(x,x)$, $x\in\Omega$ is bounded. In addition, we define

$$\hat{g}(x,y)=\tilde{g}(x,y)+\frac{pg(x,x)|x-y|^{N-p(N-2)}}{(N-p(N-2))(2N-p(N-2)-2)((N-2)\sigma_N)^{p-1}} \tag{1.21}$$

and we have for any $x\in\Omega$ that

$$\lim_{y\to x}|x-y|^{(p-2)(N-2)}\Delta\hat{g}(x,y)=-\frac{p(p-1)g(x,x)}{((N-2)\sigma_N)^{p-2}}. \tag{1.22}$$

Thus $\hat{g}(x,y)$ is regular in y for x fixed. Since $N>p(N-2)$, we take first $y=x$ in (1.21) and then the gradient and we find $\nabla_x\tilde{g}(x,x)=\nabla_x\hat{g}(x,x)$. Hence $\tilde{\phi}(x)$ is regular.

To state the next theorems we denote

$$\alpha = \frac{N}{q+1} \quad \text{and} \quad \beta = \frac{N}{p+1},$$

so the critical hyperbola (1.9) takes the form $\alpha + \beta = N - 2$.

Theorem 1.2. *Let the assumptions of Theorem 1.1 be satisfied. Then*

$$\begin{cases} \lim\limits_{\epsilon\to 0^+} \epsilon \|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{(N-2)}{\alpha}} = S^{\frac{1-pq}{p(q+1)}} \|U\|^q_{L^q(\mathbb{R}^N)} \|V\|^p_{L^p(\mathbb{R}^N)} |\phi(x_0)| \quad \text{if} \quad p > \frac{N}{N-2} \\ \lim\limits_{\epsilon\to 0^+} \epsilon \frac{\|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{(N-2)}{\alpha}}}{\log(\|u_\epsilon\|_{L^\infty(\Omega)})} = \frac{1}{\alpha} a^{\frac{N}{N-2}} S^{\frac{1-pq}{p(q+1)}} \|U\|^q_{L^q(\mathbb{R}^N)} |\phi(x_0)| \quad \text{if} \quad p = \frac{N}{N-2} \\ \lim\limits_{\epsilon\to 0^+} \epsilon \|u_\epsilon\|_{L^\infty(\Omega)}^{\frac{p(N-2)-2}{\alpha}} = S^{\frac{1-pq}{p(q+1)}} \|U\|^{q(p+1)}_{L^q(\mathbb{R}^N)} |\tilde\phi(x_0)| \quad \text{if} \quad p < \frac{N}{N-2}. \end{cases}$$

In particular taking $p = p^*$, and using (1.9) we find that $q = p^*$. We recover the results in [13, 20], that is

$$\epsilon \|u_\epsilon\|^2_{L^\infty(\Omega)} \to C \quad \text{as} \quad \epsilon \to 0, \tag{1.23}$$

for some explicitly given $C > 0$. See also [1] for the case $\Omega = B_R$.

When $N > 4$, we can take $p = 1$, and use (1.11) to find that $q = (N+4)/(N-4)$. Here we recover the result in [2, 6], where they prove that (1.23) holds for some $C > 0$.

Theorem 1.3. *Let the assumptions of Theorem 1.1 be satisfied. Then*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon(x) = \|U\|^q_{L^q(\mathbb{R}^N)} G(x, x_0), \quad \text{and} \tag{1.24}$$

$$\begin{cases} \lim\limits_{\epsilon\to 0} \|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon(x) = \|V\|^p_{L^p(\mathbb{R}^N)} G(x, x_0) \quad \text{if} \quad p > \frac{N}{N-2} \\ \lim\limits_{\epsilon\to 0} \frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)}}{\log \|u_\epsilon\|_{L^\infty(\Omega)}} u_\epsilon(x) = \frac{1}{\alpha} a^{\frac{N}{N-2}} G(x, x_0) \quad \text{if} \quad p = \frac{N}{N-2} \\ \lim\limits_{\epsilon\to 0} \|u_\epsilon\|^{\frac{1}{\alpha}(\beta + p(N-2) - N)}_{L^\infty(\Omega)} u_\epsilon(x) = \|U\|^{pq}_{L^q(\mathbb{R}^N)} \tilde G(x, x_0) \quad \text{if} \quad p < \frac{N}{N-2} \end{cases} \tag{1.25}$$

where all the convergences in $C^{1,\alpha}(w)$ with w any neighborhood of $\partial\Omega$ not containing x_0.

Remark 1.4. *For $p < \frac{N}{N-2}$, the convergence in* (1.25) *can be improved to $C^{3,\alpha}(\omega)$. See the proof of the theorem.*

Remark 1.5. *By* (2.13)*, we find that $\lim\limits_{\epsilon\to 0} \|v_\epsilon\|_{L^\infty(\Omega)} = V(0) \lim\limits_{\epsilon\to 0} \|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)}$. So, in addition, when $p = 1$ we have that*

$$\epsilon \|v_\epsilon\|^{2(N-4)/N}_{L^\infty(\Omega)} \to C V(0)^{2(N-4)/N} \quad \text{as} \quad \epsilon \to 0.$$

We can extend these results to the problem

$$-\Delta(-\Delta u_\epsilon)^{1/p} = u_\epsilon^q + \epsilon u_\epsilon \quad u_\epsilon > 0 \quad \text{in} \quad \Omega \tag{1.26}$$

$$u_\epsilon = \Delta u_\epsilon = 0 \quad \text{on} \quad \partial\Omega \tag{1.27}$$

with $\epsilon \to 0$. The existence of positive solutions for this problem can be found in [15] and [19] in the case of a ball. See [14] for related results for $p = 1$.

Theorem 1.6. *Let the assumptions of Theorem 1.1 be satisfied. Then*

$$\lim_{\epsilon\to 0^+}\epsilon\|u_\epsilon\|^{2-\frac{2}{\alpha}}_{L^\infty(\Omega)} = \|U\|^{-2}_{L^2(\mathbb{R}^N)}\|U\|^q_{L^q(\mathbb{R}^N)}\|V\|^p_{L^p(\mathbb{R}^N)}|\phi(x_0)| \quad \text{if} \quad p>\frac{N}{N-2} \quad \text{and} \quad \alpha>1,\ N>4 \tag{1.28}$$

$$\lim_{\epsilon\to 0^+}\epsilon\frac{\|u_\epsilon\|^{2-\frac{2}{\alpha}}_{L^\infty(\Omega)}}{\log(\|u_\epsilon\|_{L^\infty(\Omega)})} = \frac{1}{\alpha}a^{\frac{N}{N-2}}\|U\|^{-2}_{L^2(\mathbb{R}^N)}\|U\|^q_{L^q(\mathbb{R}^N)}|\phi(x_0)| \quad \text{if} \quad p=\frac{N}{N-2} \quad \text{and} \quad 2-\frac{2}{\alpha}=3-(\frac{N}{N-2})^2>0, \tag{1.29}$$

$$\lim_{\epsilon\to 0^+}\epsilon\|u_\epsilon\|^{2-\frac{2+N-p(N-2)}{\alpha}}_{L^\infty(\Omega)} = \|U\|^{-2}_{L^2(\mathbb{R}^N)}\|U\|^{q(p+1)}_{L^q(\mathbb{R}^N)}|\tilde{\phi}(x_0)| \quad \text{if} \quad \frac{N+4}{2(N-2)}<p<\frac{N}{N-2} \quad \text{and} \quad \alpha>\frac{2+N-p(N-2)}{2} \tag{1.30}$$

$$\lim_{\epsilon\to 0^+}\epsilon\log(\|u_\epsilon\|_{L^\infty(\Omega)}) = \frac{\|U\|^q_{L^q(\mathbb{R}^N)}\|V\|^p_{L^p(\mathbb{R}^N)}}{b^2}|\phi(x_0)| \quad \text{if} \quad N=4,\ p=q=3, \tag{1.31}$$

$$\lim_{\epsilon\to 0^+}\epsilon\log(\|u_\epsilon\|_{L^\infty(\Omega)})\|u_\epsilon\|^{\frac{3-q}{2}}_{L^\infty(\Omega)} = \frac{\|U\|^{q(p+1)}_{L^q(\mathbb{R}^N)}}{b^2}|\tilde{\phi}(x_0)| \quad \text{if} \quad p=\frac{N+4}{2(N-2)}<\frac{N}{N-2}, \quad \text{and} \quad q\le 3 \tag{1.32}$$

Note that $N>4$ (integer) is equivalent to $3-(N/(N-2))^2>0$ and also to $(N+4)/(2(N-2))<N/(N-2)$. This implies that (1.29) holds for $p=N/(N-2)$ and $N>4$, and (1.32) holds for $p=\frac{N+4}{2(N-2)}$, $q\le 3$ and $N>4$.

For example, $p=1$ gives $q+1=2N/(N-4)$ and provided that $N>8$, we get

$$\lim_{\epsilon\to 0^+}\epsilon\|u_\epsilon\|^{\frac{2(N-8)}{N-4}}_{L^\infty(\Omega)} = C_1|\tilde{\phi}(x_0)|.$$

For $N=8$ and $p=1$, we have

$$\lim_{\epsilon\to 0^+}\epsilon\log(\|u_\epsilon\|_{L^\infty(\Omega)}) = C_1|\tilde{\phi}(x_0)|.$$

2. Preliminaries

Before proving the main theorem, we need some properties of u_ϵ. Using that u_ϵ is a minimizing sequence, we have

$$\int_\Omega (\Delta u_\epsilon)^{\frac{p+1}{p}}\,dx = \int_\Omega v_\epsilon\Delta u_\epsilon\,dx = \int_\Omega u_\epsilon\Delta v_\epsilon\,dx = \int_\Omega u_\epsilon^{q_\epsilon+1}\,dx.$$

Then $[S+o(1)]\|u_\epsilon\|^{\frac{p+1}{p}}_{L^{q_\epsilon+1}(\Omega)} = \|u_\epsilon\|^{q_\epsilon+1}_{L^{q_\epsilon+1}(\Omega)}$ implies

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon^{q_\epsilon+1}\,dx = S^{\frac{pq-1}{p(q+1)}}. \tag{2.1}$$

Lemma 2.1. *The minimizing sequence* u_ϵ *of* (1.10) *is such that*

$$\|u_\epsilon\|_{L^\infty(\Omega)} \to \infty$$

moreover $\|(-\Delta u_\epsilon)^{1/p}\|_{L^\infty(\Omega)} = \|v_\epsilon\|_{L^\infty(\Omega)} \to \infty \quad as \quad \epsilon \to 0.$

Proof. If $\|u_\epsilon\|_{L^\infty(\Omega)} \to \infty$ then by regularity, we find $\|v_\epsilon\|_{L^\infty(\Omega)} \to \infty$, see [12, Theorem 3.7]. Now, assume that $\|u_\epsilon\|_{L^\infty(\Omega)} \leq M$ and $\|v_\epsilon\|_{L^\infty(\Omega)} \leq M$, by elliptic regularity, we have that

$$\|v_\epsilon\|_{C^{2+\alpha}(\bar{\Omega})} \leq M \quad \text{and} \quad \|u_\epsilon\|_{C^{2+\alpha}(\bar{\Omega})} \leq M$$

with $\alpha \in (0,1)$ and some constant M. This implies that there exists $u^*, v^* \in C^2(\bar{\Omega})$, such that

$$u_\epsilon \to u^* \quad \text{in} \quad C^2(\bar{\Omega}), \quad v_\epsilon \to v^* \quad \text{in} \quad C^2(\bar{\Omega}) \quad \text{as} \quad \epsilon \to 0.$$

Hence u^* satifies

$$0 \neq \int_\Omega (\Delta u^*)^{\frac{p+1}{p}}\, dx = S \left[\int_\Omega (u^*)^{q+1}\, dx\right]^{\frac{(p+1)}{p(q+1)}}$$

which contradicts that S can be achieved by a minimizer in a bounded domain, see [23]. In other words there exists no non trivial solution for

$$-\Delta u^* = (v^*)^p, \quad v > 0 \quad \text{in} \quad \Omega \tag{2.2}$$
$$-\Delta v^* = (u^*)^q, \quad u > 0 \quad \text{in} \quad \Omega \tag{2.3}$$
$$u^* = v^* = 0 \quad \text{on} \quad \partial\Omega \tag{2.4}$$

in a convex bounded domain, with p, q satisfying (1.9), see [18],[22]. □

For any $\epsilon > 0$, let (u_ϵ, v_ϵ) be a solution of (1.1–1.3). By the Pohozaev inequality, see [18] or [22], we have for any $\tilde{\alpha}, \tilde{\beta} \in \mathbb{R}$ that

$$\left(\frac{N}{q_\epsilon+1} - \tilde{\alpha}\right)\int_\Omega u_\epsilon^{q_\epsilon+1}\, dx + \left(\frac{N}{p+1} - \tilde{\beta}\right)\int_\Omega v_\epsilon^{p+1}\, dx \tag{2.5}$$

$$+(N-2-\tilde{\alpha}-\tilde{\beta})\int_\Omega (\nabla u_\epsilon, \nabla v_\epsilon)\, dx = -\int_{\partial\Omega} (\nabla u_\epsilon, n)(\nabla v_\epsilon, x-y)\, ds. \tag{2.6}$$

We choose $\tilde{\alpha} + \tilde{\beta} = N-2$, $\tilde{\alpha} = \alpha$ and so $\tilde{\beta} = \beta$. This implies that

$$\epsilon \int_\Omega u_\epsilon^{q_\epsilon+1}\, dx = -\int_{\partial\Omega} \frac{\partial u_\epsilon}{\partial n}\frac{\partial v_\epsilon}{\partial n}(n, x-y)\, ds. \tag{2.7}$$

Since u_ϵ becomes unbounded as $\epsilon \to 0$ we choose $\mu = \mu(\epsilon)$ and $x_\epsilon \in \Omega$ such that

$$\mu^{\alpha_\epsilon} u_\epsilon(x_\epsilon) = 1$$

where $\alpha_\epsilon = N/(q_\epsilon+1)$. Note that $\mu \to 0$ as $\epsilon \to 0$.

First we claim that x_ϵ stays away from the boundary. This is consequence of moving plane method and interior estimates [8], [11]. Let ϕ_1 the positive eigenvalue

of $(-\Delta, H^1_0(\Omega))$, normalized to $\max\limits_{x\in\Omega}\phi_1(x) = 1$. Since $p \geq 1$, multiplying by ϕ_1 we obtain

$$\lambda_1 \int\limits_\Omega u_\epsilon\phi_1 = \int\limits_\Omega v_\epsilon^p\phi_1 \geq 2\lambda_1 \int\limits_\Omega v_\epsilon\phi_1 - C\int\limits_\Omega \phi_1$$
$$\lambda_1 \int\limits_\Omega v_\epsilon\phi_1 = \int\limits_\Omega u_\epsilon^{q_\epsilon}\phi_1 \geq 2\lambda_1 \int\limits_\Omega u_\epsilon\phi_1 - C\int\limits_\Omega \phi_1$$

for some $C = C(p, q, \lambda_1) > 0$. Hence $\int\limits_\Omega u_\epsilon\phi_1 \leq (C/\lambda_1)\int\limits_\Omega \phi_1$ which implies $\int_{\Omega'} u_\epsilon \leq C(\Omega')$ with $\Omega' \subset \Omega$ and $\int_{\Omega'} v_\epsilon \leq C(\Omega')$. Using the moving planes method [11], we find that there exist $t_0\alpha > 0$ such that

$$u_\epsilon(x - t\nu) \quad \text{and} \quad v_\epsilon(x - t\nu) \quad \text{are nondecreasing for} \quad t \in [0, t_0],$$

$\nu \in \mathbb{R}^N$ with $|\nu| = 1$, and $(\nu, n(x)) \geq \alpha$ and $x \in \partial\Omega$. Therefore we can find γ, δ such that for any $x \in \{z \in \bar\Omega\colon d(z, \partial\Omega) < \delta\} = \Omega_\delta$ there exists a measurable set Γ_x with (i) $meas(\Gamma_x) \geq \gamma$, (ii) $\Gamma_x \subset \Omega \setminus \bar\Omega_{\delta/2}$, and (iii) $u_\epsilon(y) \geq u_\epsilon(x)$ and $v_\epsilon(y) \geq v_\epsilon(x)$ for any $y \in \Gamma_x$. Then for any $x \in \Omega_\delta$, we have

$$\begin{aligned} u_\epsilon(x) &\leq \frac{1}{meas(\Gamma_x)}\int\limits_{\Gamma_x} u_\epsilon(y)dy \leq \frac{1}{\gamma}\int\limits_{\Omega_\delta} u_\epsilon \leq C(\Omega_\delta), \quad \text{and} \\ v_\epsilon(x) &\leq \frac{1}{meas(\Gamma_x)}\int\limits_{\Gamma_x} v_\epsilon(y)dy \leq \frac{1}{\gamma}\int\limits_{\Omega_\delta} v_\epsilon \leq C(\Omega_\delta). \end{aligned}$$

Hence if $u_\epsilon(x_\epsilon) \to \infty$, this implies that x_ϵ will stay out of Ω_δ a neighbordhood of the boundary. This proves the claim.

Let $x_\epsilon \to x_0 \in \Omega$. We define a family of rescaled functions

$$u_{\epsilon,\mu}(y) = \mu^{\alpha_\epsilon} u_\epsilon(\mu^{1-\epsilon/2}y + x_\epsilon) \tag{2.8}$$
$$v_{\epsilon,\mu}(y) = \mu^{\beta} v_\epsilon(\mu^{1-\epsilon/2}y + x_\epsilon) \tag{2.9}$$

and find using the definitions of ϵ, α_ϵ and β, that

$$-\Delta u_{\epsilon,\mu} = v^p_{\epsilon,\mu}\mu^{\alpha_\epsilon+2-\epsilon-p\beta} = v^p_{\epsilon,\mu} \quad \text{in} \quad \Omega_\epsilon \tag{2.10}$$
$$-\Delta v_{\epsilon,\mu} = u^{q_\epsilon}_{\epsilon,\mu}\mu^{\beta+2-\epsilon-q_\epsilon\alpha_\epsilon} = u^{q_\epsilon}_{\epsilon,\mu} \quad \text{in} \quad \Omega_\epsilon \tag{2.11}$$
$$u_{\epsilon,\mu} = v_{\epsilon,\mu} = 0 \quad \text{on} \quad \partial\Omega_\epsilon. \tag{2.12}$$

By equicontinuity and using Arzela-Ascoli, we have that

$$u_{\epsilon,\mu} \to U \quad \text{and} \quad v_{\epsilon,\mu} \to V \quad \text{as} \quad \epsilon \to 0. \tag{2.13}$$

in $C^2(K)$ for any K compact in $\mathbb{R}^N$, where (U, V) satisfies (1.14)–(1.16). Now extending $u_{\epsilon,\mu}$ and $v_{\epsilon,\mu}$ by zero outside Ω_ϵ and using (2.1), by the argument in [21] or [23], we have that $u_{\epsilon,\mu} \to \bar U$ strongly (up to a subsequence) in $W^{2,\frac{p+1}{p}}(\mathbb{R}^N)$. In the limit $\bar U \in L^{q+1}(\mathbb{R}^N)$ and $\bar V := (-\Delta\bar U)^{\frac{1}{p}} \in L^{p+1}(\mathbb{R}^N)$, and they satisfy (1.14)–(1.16).

Since $p \geq 1$, the solution $(\bar{U}, \bar{V})$ is unique and radially symmetric, see [5]. In addition the radial solutions are unique [16, 23], so $\bar{U} \equiv U$ and $\bar{V} \equiv V$, consequently

$$\int_{\mathbb{R}^N} [u_{\epsilon,\mu} - U]^{q+1}(y)\, dy \to 0 \quad \int_{\mathbb{R}^N} [v_{\epsilon,\mu} - V]^{p+1}(y)\, dy \to 0. \tag{2.14}$$

Lemma 2.2. *There exists $\delta > 0$ such that*

$$\delta \leq \mu^{\epsilon} \leq 1.$$

Proof. Since $\mu \to 0$, we have $\mu^{\epsilon} \leq 1$. By (2.14), we get $\int_{B_1} u_{\epsilon,\mu}^{q_\epsilon+1}\, dx \geq M$, but

$$M \leq \int_{B_1} u_{\epsilon,\mu}^{q_\epsilon+1}\, dx = \mu^{\epsilon N/2} \int_{|y-x_\epsilon| \leq \mu^{1-\epsilon/2}} u_{\epsilon}^{q_\epsilon+1}(y)\, dy \leq \mu^{\epsilon N/2} \int_{\Omega} u_{\epsilon}^{q_\epsilon+1}(y)\, dy \tag{2.15}$$

Using the convergence (2.1), we obtain the result. □

Lemma 2.3. *There exists $K > 0$ such that the solution $(u_{\epsilon,\mu}, v_{\epsilon,\mu})$ satisfies*

$$u_{\epsilon,\mu}(y) \leq KU(y) \quad v_{\epsilon,\mu}(y) \leq KV(y) \quad \forall y \in \mathbb{R}^N. \tag{2.16}$$

We prove this lemma in section 2.3.

Lemma 2.4. *There exists a constant $C > 0$ such that*

$$\epsilon \leq C\mu^{N-2} h(\mu) \quad \textit{with} \quad h(\mu) = \begin{cases} 1 & \textit{for} \quad p > N/(N-2) \\ |\log(\mu)| & \textit{for} \quad p = N/(N-2) \\ \mu^{(p(N-2)-N)} & \textit{for} \quad p < N/(N-2). \end{cases} \tag{2.17}$$

Proof. We will establish the following

$$\int_{\partial\Omega} \frac{\partial u_\epsilon}{\partial n} \frac{\partial v_\epsilon}{\partial n}(n, x)\, dx \leq C\mu^{N-2} h(\mu)$$

and from here the result follows applying (2.7). We claim that

$$\left| \frac{\partial u_\epsilon}{\partial n} \right| \leq C\mu^{\alpha_\epsilon} \quad \left| \frac{\partial v_\epsilon}{\partial n} \right| \leq C\mu^{\beta} h(\mu)$$

In the following M is a positive constant that can vary from line to line and we shall use systematically Lemma 2.2.

For $p > N/(N-2)$, using that $-p\beta + N = \beta$, we have

$$\int_{\Omega} v_\epsilon^p(x)\, dx \leq M\mu^{-p\beta+N(1-\epsilon/2)} \int_{\mathbb{R}^N} V^p(y)\, dy \leq M\mu^{\beta}$$

and by (2.16) there exists $M > 0$ such that

$$v_\epsilon^p(x) \leq M \frac{\mu^{\beta+p(N-2)-N-p(N-2)\epsilon/2}}{|x - x_0|^{p(N-2)}}. \tag{2.18}$$

for $x \neq x_0$. Using that $\beta < \beta + p(N-2) - N$, by Lemma 5.1 we find $|\frac{\partial v_\epsilon}{\partial n}| \leq C\mu^\beta$. For u_ϵ, using that $-q_\epsilon \alpha_\epsilon + N = \alpha_\epsilon$,

$$\int_\Omega u_\epsilon^{q_\epsilon}\, dx \leq M\mu^{-q_\epsilon\alpha_\epsilon + N(1-\epsilon/2)} \int_{\mathbb{R}^N} U^q(y)\, dy \leq M\mu^{\alpha_\epsilon}$$

and by (2.16) there exist $M > 0$ such

$$u_\epsilon^{q_\epsilon}(x) \leq M \frac{\mu^{-q_\epsilon\alpha_\epsilon + q_\epsilon(N-2) - q_\epsilon(N-2)\epsilon/2}}{|x - x_0|^{q_\epsilon(N-2)}} \tag{2.19}$$

for $x \neq x_0$. Using that $\alpha_\epsilon < \alpha_\epsilon - N + q_\epsilon(N-2)$, by Lemma 5.1, we obtain $|\frac{\partial u_\epsilon}{\partial n}| \leq C\mu^{\alpha_\epsilon}$.

For $p < N/(N-2)$, we have

$$\int_\Omega v_\epsilon^p\, dx \leq M\mu^{-p\beta + p(N-2)(1-\epsilon/2)} \lim_{\mu\to 0} \frac{1}{\mu^{(p(N-2)-N)(1-\epsilon/2)}} \int_{B_{\frac{1}{\mu^{1-\epsilon/2}}}(x_\epsilon)} V^p(y)\, dy \tag{2.20}$$

$$\leq M\mu^{\beta + (p(N-2)-N)} \tag{2.21}$$

and for $x \neq x_0$, we find (2.18) for v_ϵ and for u_ϵ we have

$$\int_\Omega u_\epsilon^{q_\epsilon} \leq M\mu^{-q_\epsilon\alpha_\epsilon + N(1-\epsilon/2)} \int_{\mathbb{R}^N} U^q(y)\, dy \leq M\mu^{\alpha_\epsilon}$$

and by (2.16) there exist $M > 0$ such that

$$u_\epsilon^{q_\epsilon}(x) \leq M \frac{\mu^{-q_\epsilon\alpha_\epsilon + q_\epsilon(p(N-2)-2) - q_\epsilon(p(N-2)-2)\epsilon/2}}{|x - x_0|^{q_\epsilon(p(N-2)-2)}} \tag{2.22}$$

for $x \neq x_0$. From these estimates we prove the claim applying Lemma 5.1 and noting that $\alpha_\epsilon < \alpha_\epsilon - N + q_\epsilon(p(N-2)-2) + (p+1)\epsilon/\alpha_\epsilon$.. For the case $p = N/(N-2)$, we proceed as before noting that

$$\int_\Omega v_\epsilon^p\, dx \leq M\mu^{-p\beta + N(1-\epsilon/2)} |\log(\mu)| \lim_{\mu\to 0} \frac{1}{|\log(\mu)|} \int_{B_{\frac{1}{\mu^{1-\epsilon/2}}}(x_\epsilon)} V^p(y)\, dy \leq M|\log(\mu)|\mu^\beta$$

and for $x \neq x_0$ we have (2.18). Similarly to (2.22), we obtain that for $x \neq x_0$, there exist $M > 0$ such that

$$u_\epsilon^{q_\epsilon}(x) \leq M \frac{\mu^{-q_\epsilon\alpha_\epsilon + q_\epsilon(N-2) - q_\epsilon(N-2)\epsilon/2}}{|x - x_0|^{q_\epsilon(N-2)}} \log(|x - x_0|\mu^{-1+\epsilon/2})^{q_\epsilon}. \tag{2.23}$$

Using this and proceeding and before we prove the claim and the lemma follows. $\square$

Lemma 2.5.

$$|\mu^\epsilon - 1| = O(\mu^{N-2} h(\mu) \log \mu)$$

Proof. By the theorem of the mean $|\mu^\epsilon - 1| = |\mu^{s\epsilon} \epsilon \log \mu|$ for some $s \in (0,1)$ and therefore (2.17) gives the result. $\square$

3. Proof of Theorem 1.2 and 1.3

Proof of Theorem 1.3. We start by proving the case $p > \frac{N}{N-2}$. We have

$$
\begin{aligned}
-\Delta(\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon) &= \|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p \quad \text{in} \quad \Omega, && (3.1)\\
-\Delta(\|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon) &= \|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon} \quad \text{in} \quad \Omega, && (3.2)\\
u_\epsilon &= v_\epsilon = 0 \quad \text{on} \quad \partial\Omega. && (3.3)
\end{aligned}
$$

We integrate the right hand side of (3.1)

$$\int_\Omega \|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p \, dx = \mu^{-(p+1)\beta+N+N\epsilon/2} \int_{\Omega_\epsilon} v_{\epsilon,\mu}^p(y)\, dy.$$

But $N-(p+1)\beta = 0$, so using (2.16) by dominated convergence and Lemma 2.5, we get

$$\lim_{\epsilon\to 0} \int_\Omega \|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p \, dx = \int_{\mathbb{R}^N} V^p(y)\, dy = \|V\|_{L^p(\mathbb{R}^N)} < \infty.$$

Similarly, now using

$$\int_\Omega \|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon} \, dx = \mu^{-(q_\epsilon+1)\alpha_\epsilon+N+N\epsilon/2} \int_{\Omega_\epsilon} u_{\epsilon,\mu}^{q_\epsilon} \, dx \to \|U\|_{L^q(\mathbb{R}^N)} < \infty \tag{3.4}$$

as $\epsilon \to 0$. Also using the bound (2.16), we find

$$\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p(x) \le \frac{M\mu^{-(p+1)\beta+p(N-2)-p(N-2)\epsilon/2}}{|x-x_0|^{p(N-2)}}$$

for $x \neq x_0$ and some $M > 0$. But $-(p+1)\beta + p(N-2) > 0$ and Lemma 2.2 then $\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p(x) \to 0$ for $x \neq x_0$. Also we have

$$\|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon}(x) \le \frac{M\mu^{-(q_\epsilon+1)\alpha_\epsilon+q_\epsilon(N-2)-q_\epsilon(N-2)\epsilon/2}}{|x-x_0|^{q_\epsilon(N-2)}}.$$

for $x \neq x_0$ and some $M > 0$. But $-(q_\epsilon+1)\alpha_\epsilon + q_\epsilon(N-2) > 0$ and Lemma 2.2 then $\|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon}(x) \to 0$ for $x \neq x_0$.

From here we have

$$-\Delta(\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon) \to \|V\|^p_{L^p(\mathbb{R}^N)} \delta_{x=x_0} \quad \text{and} \quad -\Delta(\|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon) \to \|U\|^q_{L^q(\mathbb{R}^N)} \delta_{x=x_0}$$

in the sense of distributions in Ω, as $\epsilon \to 0$. Let ω be any neighborhood of $\partial\Omega$ not containing x_0. By regularity theory, see Lemma 5.1, we find

$$\|\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon\|_{C^{1,\alpha}(w)} \le C \left[\|\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p\|_{L^1(\Omega)} + \|\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} v_\epsilon^p\|_{L^\infty(w)} \right]$$

and a similar bound for $\|\|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon\|_{C^{1,\alpha}(w)}$. Consequently

$$\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon \to \|V\|^p_{L^p(\mathbb{R}^N)} G \quad \text{in} \quad C^{1,\alpha}(w) \quad \text{as} \quad \epsilon \to 0. \tag{3.5}$$

and

$$\|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon \to \|U\|^q_{L^q(\mathbb{R}^N)} G \quad \text{in} \quad C^{1,\alpha}(w) \quad \text{as} \quad \epsilon \to 0. \tag{3.6}$$

For the case $p < N/(N-2)$, we proceed as before and we have (3.4) and the bound

$$\|u_\epsilon\|_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon}(x) \le \frac{M\mu^{-(q_\epsilon+1)\alpha_\epsilon + q_\epsilon(p(N-2)-2) - q_\epsilon(p(N-2)-2)\epsilon/2}}{|x-x_0|^{q_\epsilon(p(N-2)-2)}}.$$

for $x \neq x_0$ and some $M > 0$. Using that $-(q_\epsilon+1)\alpha_\epsilon + q(p(N-2)-2) = 2(p+1) > 0$ and Lemma 2.2, we get $\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} u_\epsilon^{q_\epsilon}(x) \to 0$ for $x \neq x_0$ and hence

$$\|u_\epsilon\|_{L^\infty(\Omega)} v_\epsilon \to \|U\|^q_{L^q(\mathbb{R}^N)} G \quad \text{in} \quad C^{1,\alpha}(w) \quad \text{as} \quad \epsilon \to 0. \tag{3.7}$$

Now we claim that

$$\|u_\epsilon\|^{\frac{1}{\alpha}(\beta+p(N-2)-N)}_{L^\infty(\Omega)} u_\epsilon \to \|U\|^{pq}_{L^q(\mathbb{R}^N)} \tilde{G} \quad \text{in} \quad C^{1,\alpha}(w) \quad \text{as} \quad \epsilon \to 0. \tag{3.8}$$

We have

$$-\Delta(\|u_\epsilon\|^{\frac{1}{\alpha}(\beta+p(N-2)-N)}_{L^\infty(\Omega)} u_\epsilon) = \|u_\epsilon\|^{\frac{1}{\alpha}(\beta+p(N-2)-N)}_{L^\infty(\Omega)} v_\epsilon^p = \|u_\epsilon\|^p_{L^\infty(\Omega)} v_\epsilon^p.$$

Since the last term converges to $(\|U\|^q_{L^q(\mathbb{R}^N)} G)^p$ in $C^{1,\alpha}(\omega)$ as $\epsilon \to 0$ and $p \ge 1$, we have

$$\|u_\epsilon\|^{\frac{1}{\alpha}(\beta+p(N-2)-N)}_{L^\infty(\Omega)} u_\epsilon \to \|U\|^{pq}_{L^q(\mathbb{R}^N)} \tilde{G} \quad \text{in} \quad C^{3,\alpha}(w) \quad \text{as} \quad \epsilon \to 0.$$

For the remaining case $p = N/(N-2)$, we have as $\epsilon \to 0$, the convergence

$$\int_\Omega \frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)}}{|\log(\|u_\epsilon\|_{L^\infty(\Omega)})|} v_\epsilon^p \, dx = \frac{\mu^{-(p+1)\beta+N+N\epsilon/2}}{\alpha_\epsilon |\log(\mu)|} \int_{\Omega_\epsilon} v_{\epsilon,\mu}^p \, dy \to \frac{1}{\alpha} \lim_{r\to\infty} V(r)^{\frac{N}{N-2}} r^N = \frac{a^{\frac{N}{N-2}}}{\alpha}.$$

and the pointwise bound for $x \neq x_0$

$$\frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)}}{|\log(\|u_\epsilon\|^{\frac{\beta}{\alpha}})|} v_\epsilon^p(x) \le \frac{M\mu^{-p(N-2)\epsilon/2}}{\log(\mu)|x-x_0|^{p(N-2)}}.$$

By Lemma 2.2, $\frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)}}{|\log(\|u_\epsilon\|^{\frac{\beta}{\alpha}})|} v_\epsilon^p(x) \to 0$ for $x \neq x_0$. Writing

$$-\Delta\left(\frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty}}{|\log(\|u_\epsilon\|^{\frac{\beta}{\alpha}})|} u_\epsilon\right) = \frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty}}{|\log(\|u_\epsilon\|^{\frac{\beta}{\alpha}})|} v_\epsilon^p,$$

we observe that the last term converges to $\delta_{x=x_0}$. By Lemma 5.1, we have

$$\frac{\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty}}{|\log(\|u_\epsilon\|^{\frac{\beta}{\alpha}})|} u_\epsilon \to \frac{a^{\frac{N}{N-2}}}{\alpha} G \quad \text{in} \quad C^{1,\alpha}(w) \quad \text{as} \quad \epsilon \to 0,$$

and clearly we have (3.6) using (2.23). This completes the proof of the theorem. $\square$

Proof of Theorem 1.2. For $p > N/(N-2)$ we have

$$\epsilon\|u_\epsilon\|^{\frac{N-2}{\alpha}} \int\limits_\Omega u_\epsilon^{q_\epsilon+1}\,dx = \int\limits_{\partial\Omega} (\|u_\epsilon\|^{\frac{\beta}{\alpha}}_{L^\infty(\Omega)} \nabla u_\epsilon, n)(\|u_\epsilon\|_{L^\infty(\Omega)} \nabla v_\epsilon, n)(n, x-y)\,ds$$

By (3.5) and (3.6),

$$\lim_{\epsilon\to 0} \epsilon\|u_\epsilon\|^{\frac{N-2}{\alpha}} \int\limits_\Omega u_\epsilon^{q_\epsilon+1}\,dx = \|V\|^p_{L^p(\mathbb{R}^N)} \|U\|^q_{L^q(\mathbb{R}^N)} \int\limits_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,ds.$$

Also for the case $p < N/(N-2)$, using

$$\begin{aligned}&\epsilon\|u_\epsilon\|^{\frac{1}{\alpha}(p(N-2)-2)}_{L^\infty(\Omega)} \int\limits_\Omega u_\epsilon^{q_\epsilon+1}\,dx \\ &\quad = \int\limits_{\partial\Omega} (\|u_\epsilon\|^{\frac{1}{\alpha}(\beta+p(N-2)-N)}_{L^\infty(\Omega)} \nabla u_\epsilon, n)(\|u_\epsilon\|_{L^\infty(\Omega)} \nabla v_\epsilon, n)(n, x-y)\,ds\end{aligned}$$

and (3.8) and (3.7), we get

$$\begin{aligned}&\lim_{\epsilon\to 0} \epsilon\|u_\epsilon\|^{\frac{1}{\alpha}(p(N-2)-2)}_{L^\infty(\Omega)} \int\limits_\Omega u_\epsilon^{q_\epsilon+1}\,dx \\ &\qquad = \|U\|^{q(p+1)}_{L^q(\mathbb{R}^N)} \int\limits_{\partial\Omega} \frac{\partial \tilde G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,ds.\end{aligned}$$

The case $p = N/(N-2)$ is analogous.

The proof of the theorems follows from the next lemma. □

Lemma 3.1. *We have the following identities*

$$i) \int\limits_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,ds = -(N-2)g(x_0,x_0)$$

and

$$ii) \int\limits_{\partial\Omega} \frac{\partial \tilde G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n,, x-x_0)\,ds = -\frac{N}{q+1}\tilde g(x_0,x_0)$$

Proof. i) was proven in [3], see also [13]. To prove ii) we follow a similar procedure. From [18, 22], for any $y \in \mathbb{R}^N$, we have the following identity

$$\begin{aligned}&\int\limits_{\Omega'} \Delta u(x-y,\nabla v) + \Delta v(x-y,\nabla u) - (N-2)(\nabla u,\nabla v)dx = \\ &\int_{\partial\Omega'} \frac{\partial u}{\partial n}(x-y,\nabla v) + \frac{\partial v}{\partial n}(x-y,\nabla u) - (\nabla u,\nabla v)(x-y,n)\,ds\end{aligned}$$

where $\Omega' = \Omega \setminus B_r$ with $r > 0$. For a system $-\Delta v = 0$ and $-\Delta u = v^p$, in Ω', the identity takes the form

$$\int_{\Omega'} \frac{N}{p+1} v^{p+1} - \bar{a} v^{p+1} dx = \int_{\partial\Omega'} \frac{1}{p+1} v^{p+1}(x-y,n)\, ds$$

$$+ \int_{\partial\Omega'} \frac{\partial u}{\partial n}\left[(x-y,\nabla v) + \bar{a}v\right] + \frac{\partial v}{\partial n}\left[(x-y,\nabla u) + \bar{b}u\right] - (\nabla u, \nabla v)(x-y,n)\, ds \quad (3.9)$$

with $\bar{a} + \bar{b} = N-2$. Let $y = 0$, choose $\bar{a} = N/(p+1)$ and take $v = G(x,0)$ and $u = \tilde{G}(x,0)$. Using that $u = v = 0$ on $\partial\Omega$, and so $\nabla u = (\nabla u, n)n$ and $\nabla v = (\nabla v, n)n$ on $\partial\Omega$,, we obtain

$$\begin{aligned}\int_{\partial\Omega} \frac{\partial \tilde{G}}{\partial n}\frac{\partial \tilde{G}}{\partial n}(x,n) ds \;&=\; \int_{\partial B_r} \frac{1}{p+1} G^{p+1}(x,n) + \frac{\partial \tilde{G}}{\partial n}[(x,\nabla G) + \frac{N}{p+1} G]\, ds \\ &\quad + \int_{\partial B_r} \frac{\partial G}{\partial n}[(x,\nabla\tilde{G}) + \frac{N}{q+1}\tilde{G}] - (\nabla\tilde{G},\nabla G)(x,n)\, ds.\end{aligned}$$

Let $k = p(N-2)$ and $\Gamma = \sigma_N(N-2)$. For $|x| = r$, we have

$$\nabla\tilde{G} = -\frac{1}{\Gamma^p(N-k)}|x|^{-k}x + \nabla\tilde{g}, \quad \nabla G = -\frac{1}{\sigma_N}|x|^{-N}x + \nabla g,$$

$$\frac{\partial\tilde{G}}{\partial n} = -\frac{1}{\Gamma^p(N-k)}|x|^{1-k} + (\nabla\tilde{g}, n), \quad \frac{\partial G}{\partial n} = -\frac{1}{\sigma_N}|x|^{1-N} + (\nabla g, n)$$

$$\begin{aligned}(x,\nabla\tilde{G}) + \frac{N}{q+1}\tilde{G} \;&=\; \left(\frac{N}{(q+1)(k-2)} - 1\right)\frac{1}{\Gamma^p(N-k)}|x|^{2-k} + (x,\nabla\tilde{g}) + \frac{N}{q+1}\tilde{g} \\ (x,\nabla G) + \frac{N}{p+1}G \;&=\; \left(\frac{N}{p+1} - (N-2)\right)\frac{1}{\Gamma}|x|^{2-N} + (x,\nabla g) + \frac{N}{p+1}g\end{aligned}$$

$$(\nabla\tilde{G},\nabla G) = \frac{|x|^{-k-N+2}}{\sigma_N\Gamma^p(N-k)} - \frac{(\nabla g, x)}{\Gamma^p(N-k)}|x|^{(2-N)p} - \frac{(\nabla\tilde{g}, x)}{\sigma_N}|x|^{-N} + (\nabla\tilde{g},\nabla g)$$

and

$$\frac{1}{p+1}G^{p+1} = \frac{1}{p+1}\left[\frac{1}{\Gamma^p}|x|^{-k} - \Delta\tilde{g}\right]\left[\frac{1}{\Gamma}|x|^{2-N} + g\right]$$

From here, we check that terms with $|x|^{3-N-k}$ cancel out others integral tends to 0 since the integrand are $o(|x|^{1-N})$ and only remain one term of order $|x|^{1-N}$, giving

$$\int_{\partial\Omega} \frac{\partial\tilde{G}}{\partial n}\frac{\partial G}{\partial n}(x,n)\, ds = -\lim_{r\to 0}\frac{1}{\sigma_N r^{N-1}}\int_{\partial B_r}\frac{N}{q+1}\tilde{g} ds = -\frac{N}{q+1}\tilde{g}(0,0).$$

□

Proof of Theorem 1.1. a) The part ii) follows from Theorem 5.1,

$$\||\Delta u_\epsilon|^{\frac{1}{p}}\|_{C^{1,\alpha}(\omega)} \leq \|u_\epsilon^{q_\epsilon}\|_{L^1(\Omega)} + \|u_\epsilon^{q_\epsilon}\|_{L^\infty(\omega)}.$$

and estimates (2.19), (2.22), and (2.23). Part i) follows from

$$\|u_\epsilon\|_{C^{1,\alpha}(\omega)} \leq \|v_\epsilon^p\|_{L^1(\Omega)} + \|v_\epsilon^p\|_{L^\infty(\omega)}.$$

and estimate (2.18). Finally iii) follows combining ii) with the convergence

$$\int\limits_{\mathbb{R}^N} |\Delta u_\epsilon|^{\frac{p+1}{p}}\, dx = \int\limits_{\mathbb{R}^N} v_\epsilon^{p+1}\, dx \to \|V\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.$$

as $\epsilon \to 0$. This completes part a).

For part b), note that from (2.7), we have the vectorial equality $\int\limits_{\partial\Omega} (\nabla u_\epsilon, \nabla v_\epsilon) n\, ds = 0$. In the limit for $p \geq N/(N-2)$, we get

$$\int\limits_{\partial\Omega} (\nabla G(x,x_0), \nabla G(x,x_0)) n\, ds = 0 \tag{3.10}$$

and similarly for $p < N/(N-2)$, we obtain

$$\int\limits_{\partial\Omega} (\nabla \tilde{G}(x,x_0), \nabla G(x,x_0)) n\, ds = 0 \tag{3.11}$$

But we have the following result.

Lemma 3.2. *For every* $x_0 \in \Omega$

$$\int\limits_{\partial\Omega} (\nabla G(x,x_0), n)(\nabla G(x,x_0), n) n\, ds = -\nabla\phi(x_0) \tag{3.12}$$

and

$$\int\limits_{\partial\Omega} (\nabla \tilde{G}(x,x_0), n)(\nabla(\Delta\tilde{G}(x,x_0))^{1/p}, n) n\, ds = -\nabla\tilde{\phi}(x_0). \tag{3.13}$$

Hence combining (3.10) with (3.12), and (3.11) with (3.13), we complete the proof of part b) and the theorem is proven. □

Proof of the Lemma. Equality (3.12) was proved in [3] and [13]. To prove (3.13), by (3.9) we have

$$\int\limits_{\partial\Omega} \frac{\partial\tilde{G}}{\partial n}\frac{\partial G}{\partial n} n\, ds = \int\limits_{\partial B_r} \left\{ \frac{1}{p+1} G^{p+1} n + \frac{\partial \tilde{G}}{\partial n}\nabla G + \frac{\partial G}{\partial n}\nabla\tilde{G} - (\nabla\tilde{G}, \nabla G) n \right\} ds.$$

Using $\int\limits_{\partial B_r} n = 0$, we get

$$\int\limits_{\partial\Omega} \frac{\partial \tilde{G}}{\partial n}\frac{\partial G}{\partial n} n\, ds = \frac{1}{(p+1)r^{N-1}} \int\limits_{\partial B_r} \left\{ \frac{1}{\Gamma^p} r^{N-k-1} g - \Delta\tilde{g}\frac{1}{\Gamma} r - \Delta\tilde{g} g r^{N-1} \right\} n\, ds$$
$$+ \frac{1}{r^{N-1}} \int\limits_{\partial B_r} \{ (\nabla\tilde{g}, n)\nabla g + (\nabla g, n)\nabla\tilde{g} - (\nabla\tilde{g}, \nabla g) n \} r^{N-1}\, ds$$
$$- \frac{1}{r^{N-1}} \int\limits_{\partial B_r} \left\{ \frac{1}{\sigma_N}\nabla\tilde{g} + \frac{r^{N-k}}{\Gamma^p(N-k)}\nabla g \right\} ds. \tag{3.14}$$

We use the regular $\hat{g}(x,0)$ instead of $\tilde{g}(x,0)$. Thus

$$\nabla\hat{g}(x,0) = \nabla\tilde{g}(x,0) + \frac{pg(0,0)}{\Gamma^{p-1}(2N-k-2)}|x|^{N-k-2}x, \tag{3.15}$$

$$\Delta\hat{g}(x,0) = \Delta\tilde{g}(x,0) + \frac{pg(0,0)}{\Gamma^{p-1}}|x|^{N-k-2}. \tag{3.16}$$

But $g(x,0) = g(0,0) + (\nabla g(0,0), x) + o(|x|^2)$ and

$$\int\limits_{\partial B_r} r^{-k} g(x,0) n\, ds = \int\limits_{\partial B_1} r^{N-k-1} g(0,0) n\, ds + \int\limits_{\partial B_1} r^{N-k}(\nabla g(0,0), y) n\, ds + o(r^{N-k+1})$$

where $y = x/r$. Clearly the first integral in the r.h.s is zero and the other terms tends to zero as $r \to 0$. Hence

$$\lim_{r\to 0} \frac{1}{r^{N-1}} \int\limits_{\partial B_r} r^{N-k-1} g(x,0) n\, ds = 0. \tag{3.17}$$

We replace (3.15) and (3.16) in (3.14), to obtain an identity without $\tilde{g}$. Using the limit (3.17) and that $\hat{g}$ and g are regular, we obtain

$$\int\limits_{\partial\Omega} \frac{\partial \tilde{G}}{\partial n}\frac{\partial G}{\partial n} n\, ds = \lim_{r\to 0} \frac{1}{r^{N-1}} \int\limits_{\partial B_r} \frac{1}{\sigma_N}\nabla\hat{g}\, ds = \nabla\hat{g}(0,0) = \nabla\tilde{\phi}(0),$$

where the last equality follows by observation after Theorem 1.1. □

4. Proof of Lemma 2.3

Let us recall the problem (2.10)–(2.12),

$$-\Delta u_{\epsilon,\mu} = v_{\epsilon,\mu}^p \quad \text{in} \quad \Omega_\epsilon \tag{4.1}$$
$$-\Delta v_{\epsilon,\mu} = u_{\epsilon,\mu}^{q_\epsilon} \quad \text{in} \quad \Omega_\epsilon \tag{4.2}$$
$$u_{\epsilon,\mu} = v_{\epsilon,\mu} = 0 \quad \text{on} \quad \partial\Omega_\epsilon \tag{4.3}$$

where $\Omega_\epsilon = (\Omega - x_\epsilon)/\mu^{1-\epsilon/2}$. Let $\bar{R} > 0$. We define $\sigma(p) := 2 + N - p(N-2)$, and the scalar function

$$J(|y|) := \begin{cases} 1 \quad \text{if} \quad \sigma(p) < 2, \\ |\log(|y|/\bar{R})| \quad \text{if} \quad \sigma(p) = 2, \\ |y|^{2-\sigma(p)} \quad \text{if} \quad \sigma(p) > 2. \end{cases}$$

Note that $\sigma(p) \in [0, N)$ for $p \in (2/(N-2), (N+2)/(N-2)]$ and $\sigma(q) \leq 0$. We consider the transformations

$$z_\epsilon(y) = |y|^{2-N} v_{\epsilon,\mu}\left(\frac{y}{|y|^2}\right) \quad \text{and} \quad w_\epsilon(y) = \frac{|y|^{2-N}}{J(|y|)} u_{\epsilon,\mu}\left(\frac{y}{|y|^2}\right)$$

in Ω_ϵ^*, the image of Ω_ϵ under $x \mapsto x/|x|^2$.

The next lemma is equivalent to Lemma 2.3, using the asymptotic behaviour (1.17).

Lemma 4.1. *Let (w_ϵ, z_ϵ) solving*

$$-\Delta J(|y|) w_\epsilon = |y|^{-\sigma(p)} z_\epsilon^p \quad in \quad \Omega_\epsilon^* \tag{4.4}$$

$$-\Delta z_\epsilon = |y|^{-\sigma(q)+(q_\epsilon-q)(N-2)} [J(|y|) w_\epsilon]^{q_\epsilon} in \quad \Omega_\epsilon^* \tag{4.5}$$

$$w_\epsilon = z_\epsilon = 0 \quad on \quad \partial\Omega_\epsilon^*. \tag{4.6}$$

Then for any fixed $R \in (0, \bar{R})$, we have

$$\|w_\epsilon\|_{L^\infty(\Omega_\epsilon^R)} + \|z_\epsilon\|_{L^\infty(\Omega_\epsilon^R)} \leq C$$

where $\Omega_\epsilon^R = \Omega_\epsilon^* \cap B_R$, and $C = C(R)$ independent of $\epsilon > 0$ provided ϵ is sufficiently small.

Proof. Given $R > 0$, let w_0 and z_0 be solutions of

$$\Delta J(|y|) w_0 = 0 \quad \text{in} \quad \Omega_\epsilon^R \quad \text{and} \quad w_0 = 0, \text{ on} \quad \partial\Omega_\epsilon^* \quad w_0 = w_\epsilon \quad \text{on} \quad \partial B_R,$$

and

$$\Delta z_0 = 0 \quad \text{in} \quad \Omega_\epsilon^R \quad \text{and} \quad z_0 = 0, \text{ on} \quad \partial\Omega_\epsilon^* \quad z_0 = z_\epsilon \quad \text{on} \quad \partial B_R.$$

By the convergence in compact sets of w_ϵ and z_ϵ , see (2.13), we have $|z_\epsilon| + |\nabla z_\epsilon| + |w_\epsilon| + |\nabla w_\epsilon| \leq C$ in $|y| = R$ for C independent of ϵ. Therefore by the maximum principle, we get

$$|Jw_0| + |\nabla(Jw_0)| + |z_0| + |\nabla z_0| \leq C \quad \text{in} \quad \Omega_\epsilon^R.$$

Define $\tilde{w} = w_\epsilon - w_0$ and $\tilde{z} = z_\epsilon - z_0$. We now write

$$-\Delta J(|y|)\tilde{w} = a(y) z_\epsilon \quad \text{in} \quad \Omega_\epsilon^R \tag{4.7}$$

$$-\Delta \tilde{z} = b(y) J(|y|) w_\epsilon \quad \text{in} \quad \Omega_\epsilon^R \tag{4.8}$$

$$\tilde{w} = \tilde{z} = 0 \quad \text{on} \quad \partial\Omega_\epsilon^R \tag{4.9}$$

where $a(y) = |y|^{-\sigma(p)} z_\epsilon^{p-1}$ and $b(y) = |y|^{-\sigma(q)+(q_\epsilon-q)(N-2)} [J(|y|) w_\epsilon]^{q_\epsilon - 1}$. Clearly by the maximum principle $\tilde{w} \geq 0$ and $\tilde{z} \geq 0$.

Let $P(y) = a(y)$ and

$$Q(y) = \begin{cases} \frac{1}{M} b(y) \quad \text{for} \quad y \in B_R \setminus \bar{B}_r \\ b(y) \quad \text{for} \quad B_r \end{cases}$$

where $r \in (0, R)$ and $M > 1$ both independent of ϵ and to be determined later. Then

$$b(y)J(|y|)w_\epsilon = Q(y)J(|y|)w_\epsilon + f(y)$$

where

$$f(y) = (b(y) - Q(y))J(|y|)w_\epsilon = \begin{cases} 0 \quad \text{for} \quad y \in \Omega_\epsilon \cap B_r \\ (1 - \frac{1}{M})b(y)J(|y|)w_\epsilon \quad \text{for} \quad y \in B_R \setminus \bar{B}_r \end{cases}$$

It is clear that $f \in L^\infty(\Omega_\epsilon^R)$, in fact $\|f\|_{L^\infty(\Omega_\epsilon^R)} \leq (1 - 1/M)r^{-(2+N)}$ by using that $w_\epsilon(y) \leq Cr^{\sigma(p)-N}$ for $|y| \geq r$, when $p < N/(N-2)$, and $w_\epsilon(y) \leq Cr^{2-N}$ for $|y| \geq r$ when $p > N/(N-2)$. A similar bound is obtained for $p = N/(N-2)$. Then we transform (4.7)–(4.8) in the system

$$\begin{aligned} -\Delta J\tilde{w} &= Pz_\epsilon \quad \text{in} \quad \Omega_\epsilon^R \\ -\Delta \tilde{z} &= QJw_\epsilon + f \quad \text{in} \quad \Omega_\epsilon^R \end{aligned}$$

We define $\eta_2(y) = \chi_{w_\epsilon \leq 2\tilde{w}}(y)$ and $\eta_1(y) = \chi_{z_\epsilon \leq 2\tilde{z}}(y)$ for $y \in \Omega_\epsilon^R$, we find

$$\begin{aligned} -\Delta J\tilde{w} &\leq 2\eta_1 P\tilde{z} + f_1 \quad \text{in} \quad \Omega_\epsilon^R \\ -\Delta \tilde{z} &\leq 2\eta_2 QJ\tilde{w} + f_2 \quad \text{in} \quad \Omega_\epsilon^R \end{aligned}$$

Where $f_1 = (1 - \eta_1)Pz_\epsilon = \chi_{z_\epsilon \leq 2z_0} Pz_\epsilon \leq 2Pz_0$ and $f_2 = f + (1 - \eta_2)QJw_\epsilon$ where $(1 - \eta_2)QJw_\epsilon \leq 2QJw_0$. We write the system in the form

$$-\Delta J\tilde{w} \leq 2\eta_1 P|y|^\gamma|y|^{-\gamma}\tilde{z} + f_1 \quad \text{in} \quad \Omega_\epsilon^R, \tag{4.10}$$

$$-|y|^{-\gamma}\Delta\tilde{z} \leq 2\eta_2 Q|y|^{-\gamma}J\tilde{w} + f_2|y|^{-\gamma} \quad \text{in} \quad \Omega_\epsilon^R, \tag{4.11}$$

$$\tilde{w} = \tilde{z} = 0 \quad \text{on} \quad \partial\Omega_\epsilon^R. \tag{4.12}$$

Let $u(y) \mapsto 2\eta_2 Q|y|^{-\gamma}u(y)$ and $u(y) \mapsto 2\eta_1 P|y|^\gamma u(y)$ be the multiplication operators $\mathcal{P}$ and $\mathcal{Q}$ respectively. Note that a multiplication operator $\mathcal{C}$ with corresponding function $c(y) \in L^s(\Omega_\epsilon^R)$ is bounded from $L^{s_1}(\Omega_\epsilon^R)$ to $L^{s_2}(\Omega_\epsilon^R)$ with $1/s_2 = 1/s_1 + 1/s$.

Formally we define $-L$ the operator as $u(y) \mapsto -|y|^{-\gamma}\Delta(|y|^\gamma u(y))$. More precisely, in the appendix, we define $(-\Delta)^{-1}$ and $(-L)^{-1}$, which by the Hardy-Littlewood-Sobolev inequality are bounded, independently of ϵ, from $L^{m_1}(\Omega_\epsilon^R)$ to $L^{m_2}(\Omega_\epsilon^R)$ with $1/m_1 = 1/m_2 + 2/N$. Note that the image of these operators is a function with zero-Dirichlet boundary condition, so they are positive. Then we can write

$$J\tilde{w} \leq (-\Delta)^{-1}\mathcal{P}(-L)^{-1}(\mathcal{Q}(J\tilde{w}) + |y|^{-\gamma}f_2) + (-\Delta)^{-1}f_1.$$

Denoting by $K = (-\Delta)^{-1}\mathcal{P}(-L)^{-1}\mathcal{Q}$ and $h = K|y|^{-\gamma}f_2 + (-\Delta)^{-1}f_1$ we have

$$(I - K)J\tilde{w} \leq h$$

The proof is complete finding m large enough such that $h \in L^m(\Omega_\epsilon^R)$ and $(I - K)$ is invertible from $L^m(\Omega_\epsilon^R)$ to $L^m(\Omega_\epsilon^R)$.

We can estimate $Q(y)|y|^{-\gamma}$ in $L^{\frac{q+1}{q-1}}(\Omega_\epsilon^R)$, for $\gamma = 2\sigma(p)/(p+1) \geq 0$, and note that $\gamma = -\sigma(q)/(q+1)$ using the Sobolev Hyperbola. Since $v_{\epsilon,\mu} \to V$ in $L^{q+1}(\mathbb{R}^N)$, we

have
$$\int_{\Omega_\epsilon^*} [J(|y|)w_\epsilon(y) - V(y/|y|^2)|y|^{2-N}]^{q+1}|y|^{-\sigma(q)}\,dy \to 0 \quad \text{as} \quad \epsilon \to 0.$$
Therefore for any λ, we can take r small such that
$$\int_{\Omega_\epsilon^r} [Jw_\epsilon]^{(q+1)\frac{q_\epsilon-1}{q-1}}(y)|y|^{-\sigma(q)}\,dy \le \int_{\Omega_\epsilon^r} [Jw_\epsilon]^{(q+1)}(y)|y|^{-\sigma(q)}\,dy \le \frac{\lambda}{2C(\delta)}$$
and M large such that for all $\epsilon \le \epsilon_0$ we have
$$\begin{aligned}\int_{\Omega_\epsilon^R} [Q(s)|y|^{-\gamma}]^{\frac{q+1}{q-1}}\,dy \quad &\le \quad C(\delta)\int_{\Omega_\epsilon^r} [Jw_\epsilon]^{(q+1)\frac{q_\epsilon-1}{q-1}}|y|^{-\sigma(q)}\,dy \\ &\quad + \frac{C(\delta)}{M^{\frac{q+1}{q-1}}}\int_{B_R\setminus B_r} [Jw_\epsilon]^{(q+1)\frac{q_\epsilon-1}{q-1}}|y|^{-\sigma(q)}\,dy \le \lambda \qquad (4.13)\end{aligned}$$
where we have used $b(y) \le C(\delta)[Jw_\epsilon]^{q_\epsilon-1}$ with δ given by Lemma 2.2.

Now we show that K is bounded from $L^m(\Omega_\epsilon^R)$ to $L^m(\Omega_\epsilon^R)$.
$$\begin{aligned}\|KJ\tilde{w}\|_{L^m(\Omega_\epsilon^R)} &\le C_1\|\mathcal{P}(-L)^{-1}\mathcal{Q}J\tilde{w}\|_{L^r(\Omega_\epsilon^R)} \\ &\le C_1\||y|^\gamma 2\eta_1 P\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)}\|(-L)^{-1}\mathcal{Q}J\tilde{w}\|_{L^{r'}(\Omega_\epsilon^R)} \\ &\le C_1\||y|^\gamma 2\eta_1 P\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)} C_2\|\mathcal{Q}J\tilde{w}\|_{L^{s'}(\Omega_\epsilon^R)} \\ &\le C_1C_2\||y|^\gamma 2\eta_1 P\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)}\||y|^{-\gamma}2\eta_2 Q\|_{L^{\frac{q+1}{q-1}}(\Omega_\epsilon^R)}\|J\tilde{w}\|_{L^{m'}(\Omega_\epsilon^R)} \\ &\le \overline{C}\||y|^\gamma P\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)}\||y|^{-\gamma}Q\|_{L^{\frac{q+1}{q-1}}(\Omega_\epsilon^R)}\|J\tilde{w}\|_{L^{m'}(\Omega_\epsilon^R)}\end{aligned}$$
with $\frac{1}{r} = \frac{1}{m} + \frac{2}{N}$, so $r' > 1$ implies $m > N/(N-2)$. $\frac{1}{r} = \frac{p-1}{p+1} + \frac{1}{r'}$ and $\frac{1}{s'} = \frac{1}{r'} + \frac{2}{N}$, so condition b) in (5.1) implies $N - 2 + N/m > 2N/(p+1)$ and $s' > 1$ implies $m > (q+1)/2$ so a) in (5.1) holds since $\gamma > 0$ and $\frac{1}{s'} = \frac{q-1}{q+1} + \frac{1}{m'}$. Since
$$\frac{q-1}{q+1} + \frac{p-1}{p+1} = \frac{4}{N}, \quad \text{we have} \quad m' = m.$$
By
$$\int_{\Omega_\epsilon^*} [z_\epsilon(y) - U(y/|y|^2)|y|^{2-N}]^{p+1}|y|^{-\sigma(p)}\,dy \to 0 \quad \text{as} \quad \epsilon \to 0,$$
we deduce that $\||y|^{\gamma-\sigma(p)}z_\epsilon^{p-1}\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)} = \||y|^\gamma P\|_{L^{\frac{p+1}{p-1}}(\Omega_\epsilon^R)} \le C(\epsilon_0)$ with $C(\epsilon_0) > 0$ and for all $\epsilon \in (0, \epsilon_0)$. Since λ in (4.13) can be arbitrarely small then the norm of K is small and so $I - K\colon L^m(\Omega_\epsilon^R) \to L^m(\Omega_\epsilon^R)$ invertible for m large. We have that
$$\||y|^{-\gamma}f_2\|_{L^m(\Omega_\epsilon^R)} \le r^{-\gamma}\|f_2\|_{L^\infty(\Omega_r^R)}(\text{meas}(\Omega_r^R))^{1/m}$$
is bounded, since f_2 is zero outside Ω_r^R and
$$\|\Delta^{-1}f_1\|_{L^m(\Omega_\epsilon^R)} \le C_1\|f_2\|_{L^r(\Omega_\epsilon^R)} \le \|f_1\|_{L^\infty(\Omega_r^R)}(\text{meas}(\Omega_\mu^R))^{1/r} \le C(z_0)(\text{meas}(B_R))^{1/r}$$

This implies $\|Jw\|_{L^m(\Omega_\epsilon^R)} \le M$ for every m large, and consequently for every $m \ge 1$. (Use the w_0 to get that $\|Jw_\epsilon\|_{L^m(\Omega_\epsilon^R)} \le M$). Now we have that

$$-\Delta \tilde{z} = b(y) J w_\epsilon = |y|^{-\sigma(q)+(q-q_\epsilon)(N-2)} [J w_\epsilon]^{q_\epsilon}.$$

Since $\sigma(q) \le 0$, if we take m large such that $mq_\epsilon > N/2$ then

$$\|\tilde{z}\|_{L^\infty(\Omega_\epsilon^R)} \le \tilde{M} \quad \text{and therefore} \quad \|z_\epsilon\|_{L^\infty(\Omega_\epsilon^R)} \le M \tag{4.14}$$

for some M independent of ϵ. We study now each case of J separately. We have

$$-\Delta J w_\epsilon = |y|^{-\sigma(p)} z_\epsilon^p \quad \text{in} \quad \Omega_\epsilon^*. \tag{4.15}$$

a) In the case $J = 1$, since $\sigma(p) < 2$, using (4.14), we have $-\Delta \tilde{w}_\epsilon \in L^q(\Omega)$ for any $q \in (N/2, N/\sigma(p))$. By regularity, we get

$$\|w_\epsilon\|_{L^\infty(\Omega_\epsilon^R)} \le M.$$

b) For $J(|y|) = -\log(|y|/\bar{R}) > \log(\bar{R}/R)$, we have

$$-\Delta \tilde{w} - \frac{\nabla J}{J} \nabla \tilde{w} - \frac{\Delta J}{J} \tilde{w} = \frac{1}{J|y|^2} z_\epsilon^p \quad \text{in} \quad \Omega_\epsilon^R$$

or equivalently

$$-\Delta \tilde{w} + \frac{1}{J|y|^2}(y, \nabla \tilde{w}) + \frac{1}{J|y|^2}(N-2)\tilde{w} = \frac{1}{J|y|^2} z_\epsilon^p \quad \text{in} \quad \Omega_\epsilon^R.$$

Using (4.14), we can take $u = \tilde{w} - M$ with $M = \sup_{\epsilon>0} \sup_{y \in \Omega_\epsilon^R} z_\epsilon^p(y)/(N-2)$, and we get

$$-J|y|^2 \Delta u + (y, \nabla u) + (N-2)u \le 0 \quad \text{in} \quad \Omega_\epsilon^R.$$

Since $u = -M < 0$ on the boundary, $u \le 0$ in Ω_ϵ^R. This gives $w_\epsilon \le M$ in Ω_ϵ^R.

For the remaining case, $p < N/(N-2)$ we have

$$-\Delta \tilde{w} - \frac{\nabla J}{J} \nabla \tilde{w} - \frac{\Delta J}{J} \tilde{w} = \frac{1}{|y|^2} z_\epsilon^p \quad \text{in} \quad \Omega_\epsilon^R.$$

As before, defining $u = \tilde{w} - M$ with $M = \sup_{\epsilon>0} \sup_{y \in \Omega_\epsilon^R} z_\epsilon^p / [(\sigma(p)-2)(N-\sigma(p))]$ then

$$-|y|^2 \Delta u - (2-\sigma(p))(y, \nabla u) - (2-\sigma(p))(N-\sigma(p))u \le 0 \quad \text{in} \quad \Omega_\epsilon^R$$

Since $u = -M < 0$ on the boundary, $u \le 0$ in Ω_ϵ^R. This implies $w_\epsilon \le M$ in Ω_ϵ^R.

□

5. Appendix

Let $N > 2$. Let h and v be a function in $L^{s'}(\Omega_\epsilon^R)$. Given the Green's function G solution of $-\Delta G(x, \cdot) = \delta_x$ in Ω_ϵ^R, $G(x, \cdot) = 0$ on $\partial\Omega_\epsilon^R$, we define

$$(-\Delta)^{-1} h(\xi) = \int_{\Omega_\epsilon^R} G(x, \xi) h(x)\, dx \quad \xi \in \Omega_\epsilon^R.$$

and

$$(-L)^{-1}v(\xi) = |\xi|^{-\gamma} \int\limits_{\Omega_\epsilon^R} G(x,\xi)|x|^{\gamma} v(x)\, dx \quad \xi \in \Omega_\epsilon^R.$$

Note that G is positive, so both operators are positive. We know that $(-\Delta)^{-1}$ is bounded, independently of ϵ, from $L^{s'}(\Omega_\epsilon^R)$ to $L^{r'}(\Omega_\epsilon^R)$ with $1/r' = 1/s' - 2/N$. Next we prove the same result for $(-L)^{-1}$. By the weighted Hardy-Littlewood-Sobolev inequality [5, 17], for $|\xi|^{-\gamma} f \in L^{s'}(\Omega_\epsilon^R)$, we have that

$$\|\xi^{-\gamma}(-\Delta)^{-1} f\|_{L^{r'}(\Omega_\epsilon^R)} \le 2\||\xi|^{-\gamma} \int\limits_{\Omega_\epsilon^R} \frac{C}{|x-\xi|^{N-2}} f(x)\, dx\|_{L^{r'}(\Omega_\epsilon^R)} \le C\||\xi|^{-\gamma} f\|_{L^{s'}(\Omega_\epsilon^R)}$$

for $1 < s' < r' < \infty$, with $1/r' = 1/s' - 2/N$ and

$$a) \quad -\gamma < N(1-1/s') = N-2-N/r' \quad \text{and} \quad b) \quad \gamma < N/r'. \tag{5.1}$$

In other words, for any $v \in L^{s'}(\Omega_\epsilon^R)$, we have

$$\begin{aligned} \|(-L)^{-1}v\|_{L^{r'}(\Omega_\epsilon^R)} &= \||\xi|^{-\gamma}(-\Delta)^{-1}|x|^{\gamma} v\|_{L^{r'}(\Omega_\epsilon^R)} \\ &\le 2\||\xi|^{-\gamma} \int\limits_{\Omega_\epsilon^R} \frac{C}{|x-\xi|^{N-2}} |x|^{\gamma} v(x)\, dx\|_{L^{r'}(\Omega_\epsilon^R)} \\ &\le C\|v\|_{L^{s'}(\Omega_\epsilon^R)}. \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let u solve*

$$\begin{cases} -\Delta u = f & in \quad \Omega \subset \mathbb{R}^N \\ u = 0 & on \quad \partial\Omega \end{cases}$$

Let ω be a neighborhood of $\partial\Omega$. Then

$$\|u\|_{W^{1,q}(\Omega)} + \|\nabla u\|_{C^{0,\alpha}(\omega')} \le C(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)})$$

for $q < N/(N-1)$, $\alpha \in (0,1)$ and $\omega' \subset \omega$ is a strict subdomain of ω.

Acknowledgements

The author thank Prof. J. Dávila for many useful discussions.

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